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Δ-Sobolev orthogonal polynomials of Meixner type: asymptotics and limit relation

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Abstract

Let $\{Q_n(x)\}_n$ be the sequence of monic polynomials orthogonal with respect to the Sobolev-type inner product

$$\langle p(x), r(x)\rangle_S = \langle \boldsymbol{u}_0, p(x)r(x)\rangle + \lambda\langle \boldsymbol{u}_1, (\Delta p)(x)(\Delta r)(x)\rangle,$$

where $\lambda \geqslant 0$, $(\Delta f)(x) = f(x+1) - f(x)$ denotes the forward difference operator and $(\boldsymbol{u}_0, \boldsymbol{u}_1)$ is a Δ-coherent pair of positive-definite linear functionals being $\boldsymbol{u}_1$ the Meixner linear functional. In this paper, relative asymptotics for the $\{Q_n(x)\}_n$ sequence with respect to Meixner polynomials on compact subsets of $\mathbb{C}\backslash[0, +\infty)$ is obtained. This relative asymptotics is also given for the scaled polynomials. In both cases, we deduce the same asymptotics as we have for the self-Δ-coherent pair, that is, when $\boldsymbol{u}_0 = \boldsymbol{u}_1$ is the Meixner linear functional. Furthermore, we establish a limit relation between these orthogonal polynomials and the Laguerre–Sobolev orthogonal polynomials which is analogous to the one existing between Meixner and Laguerre polynomials in the Askey scheme.

Keywords: Orthogonal polynomials; Sobolev orthogonal polynomials; Meixner polynomials; Δ-coherent pairs; Asymptotics; Linear functionals

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1. Introduction

The study of polynomials orthogonal with respect to an inner product involving differences was started in two papers [7,8] by Bavinck. There, the inner product

$$\langle p, q\rangle = \int_{\mathbb{R}} p(t)q(t)\,\mathrm{d}\mu(t) + \lambda(\Delta p)(c)(\Delta q)(c) \tag{1.1}$$

was introduced, where p, q are polynomials with real coefficients, $c \in \mathbb{R}$, μ is a distribution function with infinite support such that μ has no points of increase in the interval $(c, c+1)$, $\lambda \in \mathbb{R}_+$ and $(\Delta p)(c) = p(c+1) - p(c)$ denotes the forward difference operator.

Some algebraic and analytic results for the polynomials orthogonal with respect to (1.1) were obtained in [7,8], with special emphasis on the location of their zeros. Furthermore, when μ is the Meixner weight function and $c = 0$, spectral properties were deduced. Later on, in [9] the authors obtained a difference operator of infinite order for which these orthogonal polynomials (called Sobolev-type Meixner polynomials) are eigenfunctions. The name Sobolev-type is justified from the analogy with the case

$$\langle p, q\rangle = \int_{\mathbb{R}} p(t)q(t)\,\mathrm{d}\mu(t) + Mp'(c)q'(c), \tag{1.2}$$

which has been widely considered in the literature (for example, see the survey in Sobolev polynomials [13]). Note that (1.2) can be considered as a limit case of (1.1). Later on, under the influence of the developments in the so-called continuous case, i.e., inner products of the form

$$\langle p, q\rangle = \int_{\mathbb{R}} p(t)q(t)\,\mathrm{d}\mu_0(t) + \lambda \int_{\mathbb{R}} p'(t)q'(t)\,\mathrm{d}\mu_1(t),$$

where μ_0 and μ_1 are nonatomic measures satisfying some extra conditions (the so-called coherence, see [12,14]), the research is focused on the analysis of polynomials orthogonal with respect to the inner product

$$\langle p, q\rangle = \int_{\mathbb{R}} p(t)q(t)\,\mathrm{d}\mu_0(t) + \lambda \int_{\mathbb{R}} (\Delta p)(t)(\Delta q)(t)\,\mathrm{d}\mu_1(t), \tag{1.3}$$

where μ_0, μ_1 are measures with a countable set as its support. In that case, the concept of Δ-coherent pair was introduced as a discrete analogue of the continuous case (see [4]) and a classification of such Δ-coherent pairs was given (see [2]). One of the measures μ_0, μ_1 must be classical, i.e., correspond to Charlier, Kravchuk, Meixner, or Hahn polynomials. In particular, the Meixner linear functional is self-Δ-coherent. If $\mu_0 = \mu_1$ is the Meixner weight function, the relative asymptotics for these polynomials orthogonal with respect to (1.3) in terms of the Meixner polynomials has been analyzed in [5] as well as some analytic properties of their zero distribution (see [3]).

In this paper we study the asymptotic properties for polynomials orthogonal with respect to a Δ-Sobolev inner product built from a Δ-coherent pair of measures (μ_0, μ_1) of type I, that is, assuming that μ_1 is the Meixner weight function and therefore, according to the classification of the Δ-coherent pairs given in [2], μ_0 is a polynomial modification of degree one of the measure μ_1. We will establish that this polynomial modification does not influence the asymptotic behavior of the Δ-Sobolev orthogonal polynomials. Another goal of this paper is to show that one family of continuous Sobolev orthogonal polynomials can be seen as a limit of the Δ-Sobolev polynomials introduced in this work. In this way, we obtain some results contained in [15] for the Laguerre–Sobolev orthogonal polynomials.

The structure of the paper is the following: In Section 2, we introduce very well-known properties of Meixner polynomials which will be very useful along this paper and some notions about Δ-coherence. In Section 3, the outer relative asymptotics and the outer Plancherel–Rotach type asymptotics for polynomials orthogonal with respect to the inner product (1.3) when (μ_0, μ_1) is a Δ-coherent pair of type I in terms of Meixner polynomials are deduced. Finally, in Section 4, we establish a limit relation between these orthogonal polynomials and the Laguerre–Sobolev orthogonal polynomials which is analogous to the one existing between Meixner and Laguerre polynomials in the Askey scheme and we recover some results given in [15] for Laguerre–Sobolev orthogonal polynomials.

2. Basic definitions and notations

2.1. Δ-coherent pairs of linear functionals

Let $\mathbb{P}$ be the linear space of polynomials with complex coefficients and let $\mathbb{P}'$ be its algebraic dual space. We denote $\langle \boldsymbol{u}, p\rangle$ the duality bracket for $\boldsymbol{u} \in \mathbb{P}'$ and $p \in \mathbb{P}$, and $(\boldsymbol{u})_n = \langle \boldsymbol{u}, x^n\rangle$ with $n \geqslant 0$ are the canonical moments of $\boldsymbol{u}$.

Definition 2.1. A linear functional $\boldsymbol{u}$ is said to be quasi-definite if all the principal submatrices $H_k = [(\boldsymbol{u})_{i+j}]_{i,j=0}^{k}$, $k \geqslant 0$, of the Hankel moment matrix associated with $\boldsymbol{u}$ are nonsingular.

Given a quasi-definite linear functional $\boldsymbol{u}$, there exists a family of monic polynomials $\{P_n(x)\}_{n=0}^{\infty}$ orthogonal with respect to $\boldsymbol{u}$, i.e. $P_n(x) = x^n +$ terms of lower degree, for every $n \geqslant 0$, and $\langle \boldsymbol{u}, P_n P_m\rangle = \Gamma_n \delta_{n,m}$, $\Gamma_n \neq 0$, for every $n, m \geqslant 0$. Such a sequence is said to be a monic orthogonal polynomial sequence (MOPS) associated with the linear functional $\boldsymbol{u}$.

Next, we introduce the concept of positive-definite linear functional [10, p. 13].

Definition 2.2. A linear functional $\boldsymbol{u}$ is said to be positive-definite if its moments are all real and $\det(H_k) > 0$, for every $k \geqslant 0$.

Definition 2.3. Given a complex number c, the Dirac functional $\boldsymbol{\delta}(x-c)$ is defined by

$$\langle \boldsymbol{\delta}(x-c), p\rangle := p(c), \quad \text{for every } p \in \mathbb{P}.$$

Definition 2.4. Let $\boldsymbol{u}$ be a linear functional and p be a fixed polynomial. We define the linear functional $p(x)\boldsymbol{u}$ as follows:

$$\langle p\boldsymbol{u}, q\rangle := \langle \boldsymbol{u}, pq\rangle, \quad \text{for every } q \in \mathbb{P}.$$

For each complex number c we introduce the linear functional $(x-c)^{-1}\boldsymbol{u}$ such that

$$\langle (x-c)^{-1}\boldsymbol{u}, q\rangle := \left\langle \boldsymbol{u}, \frac{q(x)-q(c)}{x-c}\right\rangle, \quad \text{for every } q \in \mathbb{P}.$$

Note that

$$(x-c)^{-1}((x-c)\boldsymbol{u}) = \boldsymbol{u} - (\boldsymbol{u})_0 \boldsymbol{\delta}(x-c),$$

while $(x-c)((x-c)^{-1}\boldsymbol{u}) = \boldsymbol{u}$.

Definition 2.5. The forward difference operator Δ and the backward difference operator ∇ are defined by

$$(\Delta f)(x) := f(x+1) - f(x), \quad (\nabla f)(x) := f(x) - f(x-1).$$

Definition 2.6. For $\boldsymbol{u} \in \mathbb{P}'$, we introduce the linear functional $\Delta\boldsymbol{u}$ as

$$\langle \Delta\boldsymbol{u}, p\rangle = -\langle \boldsymbol{u}, \Delta p\rangle, \quad \text{for every } p \in \mathbb{P}.$$

Definition 2.7. A linear functional $\boldsymbol{u}$ is said to be a classical discrete linear functional if $\boldsymbol{u}$ is quasi-definite and there exist polynomials ϕ and ψ, with $\deg(\phi) \leqslant 2$ and $\deg(\psi) = 1$ such that

$$\Delta[\phi\boldsymbol{u}] = \psi\boldsymbol{u}. \tag{2.1}$$

The corresponding MOPS associated with $\boldsymbol{u}$ is said to be a classical discrete MOPS.

The Meixner linear functional $\boldsymbol{u}^{(\gamma,\mu)}$, defined by

$$\langle \boldsymbol{u}^{(\gamma,\mu)}, p\rangle := \sum_{x=0}^{\infty} p(x) \frac{\mu^x \Gamma(x+\gamma)\,(1-\mu)^{\gamma}}{\Gamma(\gamma)\Gamma(x+1)}, \quad \gamma > 0, \quad 0 < \mu < 1, \quad p \in \mathbb{P} \tag{2.2}$$

is a classical discrete linear functional since it satisfies the distributional equation (2.1) with

$$\phi(x) := \mu(x+\gamma), \quad \psi(x) := \gamma\mu - x(1-\mu).$$

Definition 2.8. Let $\boldsymbol{u}_0$ and $\boldsymbol{u}_1$ be two quasi-definite linear functionals, and let $\{P_n(x)\}_n$ and $\{T_n(x)\}_n$ be the MOPS associated with $\boldsymbol{u}_0$ and $\boldsymbol{u}_1$, respectively. We say that $(\boldsymbol{u}_0, \boldsymbol{u}_1)$ is a Δ-coherent pair of linear functionals if

$$T_n(x) = \frac{(\Delta P_{n+1})(x)}{n+1} - \sigma_n \frac{(\Delta P_n)(x)}{n}, \quad n \geqslant 1, \tag{2.3}$$

where $\{\sigma_n\}_n$ is a sequence of nonzero complex numbers.

In [4] we have proved that if $(\boldsymbol{u}_0, \boldsymbol{u}_1)$ is a Δ-coherent pair of linear functionals, then at least one of them must be a classical discrete linear functional (Charlier, Meixner).

2.2. Δ-coherent pairs of Meixner type

Definition 2.9. Let $(\boldsymbol{u}_0, \boldsymbol{u}_1)$ be a Δ-coherent pair of linear functionals. If $\boldsymbol{u}_0$ or $\boldsymbol{u}_1$ is the Meixner linear functional $\boldsymbol{u}^{(\gamma,\mu)}$ defined in (2.2), then $(\boldsymbol{u}_0, \boldsymbol{u}_1)$ is said to be a Δ-coherent pair of Meixner type.

Furthermore, we deduced in [2] the following:

Proposition 2.10. *Let* $(\boldsymbol{u}_0, \boldsymbol{u}_1)$ *be a* Δ*-coherent pair of positive-definite linear functionals of Meixner type.*

(1) *If* $\boldsymbol{u}_1$ *is the Meixner linear functional* $\boldsymbol{u}^{(\gamma,\mu)}$, *then*

(a) *If* $\gamma > 1$, *then*

$$\boldsymbol{u}_0 = (1-\mu)\left(\frac{x+a}{\gamma-1}\right)\boldsymbol{u}^{(\gamma-1,\mu)} = \boldsymbol{u}^{(\gamma,\mu)} + \frac{(1-\mu)(1-\gamma+a)}{\gamma-1}\boldsymbol{u}^{(\gamma-1,\mu)}, \quad a \geqslant 0. \tag{2.4}$$

(b) *If* $\gamma = 1$, *then* $\boldsymbol{u}_0 = \boldsymbol{u}^{(1,\mu)} + K\boldsymbol{\delta}(x)$, *with* $K \geqslant 0$.

(c) *If* $0 < \gamma < 1$, *then* $\boldsymbol{u}_0 = \boldsymbol{u}^{(\gamma,\mu)}$.

(2) *If* $\boldsymbol{u}_0$ *is the Meixner linear functional* $\boldsymbol{u}^{(\gamma,\mu)}$, *then*

$$\boldsymbol{u}_1 = \frac{\gamma}{(1-\mu)(x+a)}\boldsymbol{u}^{(\gamma+1,\mu)} + K\boldsymbol{\delta}(x+a), \quad K > 0, \ \ a \geqslant 0. \tag{2.5}$$

2.3. Monic Meixner polynomials

Monic Meixner orthogonal polynomials denoted by $\{M_n^{(\gamma,\mu)}(x)\}_n$ are the polynomial solution of a second-order linear difference equation of hypergeometric type [11,16]

$$\sigma(x)(\Delta\nabla y)(x) + \tau(x)(\Delta y)(x) + \lambda_n y(x) = 0,$$

$$\sigma(x) := x, \quad \tau(x) := \gamma\mu - x(1-\mu), \quad \lambda_n := n(1-\mu). \tag{2.6}$$

These polynomials $\{M_n^{(\gamma,\mu)}(x)\}_n$ are orthogonal on $\mathbb{N} \cup \{0\}$ with respect to the linear functional (2.2).

For monic Meixner orthogonal polynomials we get [6,11,16].

2.3.1. Three-term recurrence relation

$$xM_n^{(\gamma,\mu)}(x) = M_{n+1}^{(\gamma,\mu)}(x) + B_n^{(\gamma,\mu)}M_n^{(\gamma,\mu)}(x) + C_n^{(\gamma,\mu)}M_{n-1}^{(\gamma,\mu)}(x), \quad n \geqslant 1, \tag{2.7}$$

$$B_n^{(\gamma,\mu)} = \frac{\gamma\mu + n(1+\mu)}{1-\mu}, \quad C_n^{(\gamma,\mu)} = \frac{\mu n(\gamma+n-1)}{(1-\mu)^2} \tag{2.8}$$

with the initial conditions $M_{-1}^{(\gamma,\mu)}(x) := 0$ and $M_0^{(\gamma,\mu)}(x) := 1$. Furthermore, for $n \geqslant 0$,

$$M_n^{(\gamma,\mu)}(0) = \left(\frac{\mu}{\mu-1}\right)^n (\gamma)_n, \quad M_n^{(1,\mu)}(0) + n\frac{\mu}{1-\mu}M_{n-1}^{(1,\mu)}(0) = 0, \tag{2.9}$$

where $(a)_s$ denotes the Pochhammer symbol, $(a)_0 = 1$, $(a)_s = a(a+1)\cdots(a+s-1)$, $s \geqslant 1$.

2.3.2. Squared norm

From (2.8)

$$k_n^{(\gamma,\mu)} := \langle \boldsymbol{u}^{(\gamma,\mu)}, (M_n^{(\gamma,\mu)}(x))^2 \rangle = \frac{n!(\gamma)_n \mu^n}{(1-\mu)^{2n}}, \quad n \geqslant 0. \tag{2.10}$$

The following relations can be easily derived from the definition of $k_n^{(\gamma,\mu)}$:

$$k_0^{(\gamma,\mu)} = 1, \quad k_n^{(\gamma,\mu)} = \frac{(\gamma+n-1)\mu n}{(1-\mu)^2} k_{n-1}^{(\gamma,\mu)}, \quad n \geqslant 1. \tag{2.11}$$

2.3.3. Difference representations

We have

$$M_n^{(\gamma,\mu)}(x) = \frac{(\Delta M_{n+1}^{(\gamma,\mu)})(x)}{n+1} + \frac{\mu}{1-\mu}(\Delta M_n^{(\gamma,\mu)})(x), \quad n \geqslant 0. \tag{2.12}$$

The following relation between two different sequences of Meixner polynomials holds:

$$M_n^{(\gamma+1,\mu)}(x) = \frac{(\Delta M_{n+1}^{(\gamma,\mu)})(x)}{n+1}, \quad n \geqslant 0. \tag{2.13}$$

From the above relation and (2.12), we get

$$M_n^{(\gamma-1,\mu)}(x) = M_n^{(\gamma,\mu)}(x) + n\frac{\mu}{1-\mu} M_{n-1}^{(\gamma,\mu)}(x), \quad n \geqslant 0, \tag{2.14}$$

which is valid for $\gamma > 1$.

2.3.4. Asymptotic property

From (2.7) and using Poincaré's Theorem the relative asymptotics

$$\lim_{n\to\infty} \frac{M_{n+1}^{(\gamma,\mu)}(x)}{n M_n^{(\gamma,\mu)}(x)} = \frac{1}{\mu-1}, \tag{2.15}$$

holds uniformly on compact subsets of $\mathbb{C}\setminus[0,+\infty)$.

3. Asymptotics of Δ-Meixner–Sobolev orthogonal polynomials of type I

We shall denote $\{Q_n(x;\gamma,\mu;\lambda,K,a) \equiv Q_n(x)\}_n$ the sequence of monic polynomials orthogonal with respect to the Sobolev type inner product

$$\langle p(x), r(x)\rangle_{\mathrm{S}}^{\mathrm{M}} := \langle \boldsymbol{u}_0, p(x)r(x)\rangle + \lambda\langle \boldsymbol{u}_1, (\Delta p)(x)(\Delta r)(x)\rangle, \tag{3.1}$$

where $(\boldsymbol{u}_0, \boldsymbol{u}_1)$ be a Δ-coherent pair of linear functionals with $\boldsymbol{u}_1 = \boldsymbol{u}^{(\gamma,\mu)}$ and we shall refer to this as Δ-coherent pairs of Meixner type I. Moreover, we shall denote

$$\tilde{k}_n^{\mathrm{M}} := \langle Q_n(x), Q_n(x)\rangle_{\mathrm{S}}^{\mathrm{M}}, \quad n \geqslant 0. \tag{3.2}$$

We can summarize the asymptotic behavior of polynomials orthogonal with respect to (3.1) in the following:

Theorem 3.1 (*Relative asymptotics*). *Let* $(\boldsymbol{u}_0, \boldsymbol{u}_1)$ *be a* Δ*-coherent pair of linear functionals of Meixner type I. Let us denote* $\{Q_n(x)\}_n$ *the MOPS with respect to* (3.1) *and*

$$\eta := \frac{\mu(1+\mu) + \lambda(1-\mu)^2 + \sqrt{(\mu(1+\mu)+\lambda(1-\mu)^2)^2 - 4\mu^3}}{2\mu}. \tag{3.3}$$

The following limit relation holds:

$$\lim_{n\to\infty} \frac{Q_n(x)}{M_n^{(\gamma,\mu)}(x)} = \frac{\eta(1-\mu)}{\eta-\mu}, \tag{3.4}$$

uniformly on compact subsets of $\mathbb{C}\backslash[0,+\infty)$.

In order to prove the above theorem we need some analytic and algebraic results.

Lemma 3.2. *Let* $(\boldsymbol{u}_0, \boldsymbol{u}_1)$ *be a* Δ*-coherent pair of linear functionals, with* $\boldsymbol{u}_1 = \boldsymbol{u}^{(\gamma,\mu)}$. *Then,*

$$M_n^{(\gamma,\mu)}(x) + n\frac{\mu}{1-\mu}M_{n-1}^{(\gamma,\mu)}(x) = Q_n(x) + n\frac{\mu}{1-\mu}\frac{k_{n-1}^{(\gamma,\mu)}}{\tilde{k}_{n-1}^{M}}Q_{n-1}(x), \quad n\geqslant 1, \tag{3.5}$$

where $k_n^{(\gamma,\mu)}$ *and* $\tilde{k}_n^M$ *are given in* (2.10) *and* (3.2), *respectively, and* $Q_0(x)=1$.

Proof. (1) If $\gamma>1$, then $\boldsymbol{u}_0$ is given in (2.4). If we consider the expansion

$$M_n^{(\gamma,\mu)}(x) + n\frac{\mu}{1-\mu}M_{n-1}^{(\gamma,\mu)}(x) = Q_n(x) + \sum_{i=0}^{n-1} f_{i,n}Q_i(x),$$

then, from (2.12) and (2.14), we get, for $0\leqslant i\leqslant n-1$,

$$\begin{aligned}
f_{i,n} &= \frac{1}{\tilde{k}_i^M}\langle M_n^{(\gamma,\mu)}(x) + n\frac{\mu}{1-\mu}M_{n-1}^{(\gamma,\mu)}(x),\, Q_i(x)\rangle_{\mathrm{S}}^{\mathrm{M}} \\
&= \frac{1}{\tilde{k}_i^M}\left\{\left\langle \boldsymbol{u}_0, \left(M_n^{(\gamma,\mu)}(x) + n\frac{\mu}{1-\mu}M_{n-1}^{(\gamma,\mu)}(x)\right)Q_i(x)\right\rangle\right. \\
&\quad \left. + \lambda\left\langle \boldsymbol{u}^{(\gamma,\mu)}, \left((\Delta M_n^{(\gamma,\mu)})(x) + n\frac{\mu}{1-\mu}(\Delta M_{n-1}^{(\gamma,\mu)})(x)\right)(\Delta Q_i)(x)\right\rangle\right\} \\
&= \frac{1}{\tilde{k}_i^M}\langle \boldsymbol{u}_0, M_n^{(\gamma-1,\mu)}(x)Q_i(x)\rangle.
\end{aligned}$$

Taking into account (2.4), $f_{i,n}=0$ for $0\leqslant i\leqslant n-2$. Finally, if $i=n-1$,

$$
\begin{aligned}
f_{n-1,n} &= \frac{\langle \boldsymbol{u}_0, M_n^{(\gamma-1,\mu)}(x)Q_{n-1}(x)\rangle}{\tilde{k}_{n-1}^M}\\
&= \frac{1}{\tilde{k}_{n-1}^M}\left\langle \boldsymbol{u}^{(\gamma,\mu)} + \frac{(1-\gamma+a)(1-\mu)}{\gamma-1}\boldsymbol{u}^{(\gamma-1,\mu)}, M_n^{(\gamma-1,\mu)}(x)Q_{n-1}(x)\right\rangle\\
&= \frac{1}{\tilde{k}_{n-1}^M}\langle \boldsymbol{u}^{(\gamma,\mu)}, M_n^{(\gamma-1,\mu)}(x)Q_{n-1}(x)\rangle\\
&= \frac{1}{\tilde{k}_{n-1}^M}\left\langle \boldsymbol{u}^{(\gamma,\mu)}, \left(M_n^{(\gamma,\mu)}(x) + n\frac{\mu}{1-\mu}M_{n-1}^{(\gamma,\mu)}(x)\right)Q_{n-1}(x)\right\rangle\\
&= n\frac{\mu}{1-\mu}\frac{k_{n-1}^{(\gamma,\mu)}}{\tilde{k}_{n-1}^M}.
\end{aligned}
$$

(2) If $\gamma=1$, then $\boldsymbol{u}_0=\boldsymbol{u}^{(1,\mu)}+K\boldsymbol{\delta}(x)$, with $K\geqslant 0$. Thus

$$
M_n^{(1,\mu)}(x) + n\frac{\mu}{1-\mu}M_{n-1}^{(1,\mu)}(x) = Q_n(x) + \sum_{i=0}^{n-1} g_{i,n}Q_i(x), \quad n\geqslant 1,
$$

and the coefficients $g_{i,n}$, for $0\leqslant i\leqslant n-1$, can be computed by using (2.12) and (2.9). Indeed

$$
\begin{aligned}
g_{i,n} &= \frac{\langle M_n^{(1,\mu)}(x) + n\frac{\mu}{1-\mu}M_{n-1}^{(1,\mu)}(x), Q_i(x)\rangle_{\mathrm{S}}^{\mathrm{M}}}{\tilde{k}_i^{\mathrm{M}}}\\
&= \frac{1}{\tilde{k}_i^M}\left\{\left\langle \boldsymbol{u}_0, \left(M_n^{(1,\mu)}(x) + n\frac{\mu}{1-\mu}M_{n-1}^{(1,\mu)}(x)\right)Q_i(x)\right\rangle\right.\\
&\quad \left. + \lambda\left\langle \boldsymbol{u}_1, \left((\Delta M_n^{(1,\mu)})(x) + n\frac{\mu}{1-\mu}(\Delta M_{n-1}^{(1,\mu)})(x)\right)(\Delta Q_i)(x)\right\rangle\right\}\\
&= \frac{1}{\tilde{k}_i^M}\left\{\left\langle \boldsymbol{u}^{(1,\mu)}, \left(M_n^{(1,\mu)}(x) + n\frac{\mu}{1-\mu}M_{n-1}^{(1,\mu)}(x)\right)Q_i(x)\right\rangle\right.\\
&\quad + K\left(M_n^{(1,\mu)}(0) + n\frac{\mu}{1-\mu}M_{n-1}^{(1,\mu)}(0)\right)Q_i(0)\\
&\quad \left. + \lambda\langle \boldsymbol{u}^{(1,\mu)}, nM_{n-1}^{(1,\mu)}(x)\Delta Q_i(x)\rangle\right\}\\
&= \frac{1}{\tilde{k}_i^M}\left\langle \boldsymbol{u}^{(1,\mu)}, \left(M_n^{(1,\mu)}(x) + n\frac{\mu}{1-\mu}M_{n-1}^{(1,\mu)}(x)\right)Q_i(x)\right\rangle.
\end{aligned}
$$

Thus $g_{i,n}=0$ for $0\leqslant i\leqslant n-2$. Furthermore,

$$
g_{n-1,n} = n\frac{\mu}{1-\mu}\frac{k_{n-1}^{(1,\mu)}}{\tilde{k}_{n-1}^M}.
$$

(3) If $0<\gamma<1$, then $\boldsymbol{u}_0=\boldsymbol{u}_1=\boldsymbol{u}^{(\gamma,\mu)}$ and (3.5) was already obtained in [3]. $\square$

Lemma 3.3. *The following recurrence relation for* $\tilde{k}_n^M$ *holds*:

(1) *If* $\gamma > 1$, *then for* $n \geqslant 1$

$$\tilde{k}_n^M = k_n^{(\gamma,\mu)} + n^2\left(\left(\frac{\mu}{1-\mu}\right)^2 + \lambda\right)k_{n-1}^{(\gamma,\mu)} + \frac{(1-\gamma+a)(1-\mu)}{\gamma-1}k_n^{(\gamma-1,\mu)} - \left(n\frac{\mu}{1-\mu}\right)^2\frac{(k_{n-1}^{(\gamma,\mu)})^2}{\tilde{k}_{n-1}^M} \tag{3.6}$$

with the initial condition

$$\tilde{k}_0^M := \frac{(\gamma-1)\mu + a(1-\mu)}{\gamma-1}.$$

(2) *If* $0 < \gamma \leqslant 1$, *then for* $n \geqslant 1$

$$\tilde{k}_n^M = k_n^{(\gamma,\mu)} + n^2\left(\left(\frac{\mu}{1-\mu}\right)^2 + \lambda\right)k_{n-1}^{(\gamma,\mu)} - \left(n\frac{\mu}{1-\mu}\right)^2\frac{(k_{n-1}^{(\gamma,\mu)})^2}{\tilde{k}_{n-1}^M} \tag{3.7}$$

with the initial condition

$$\tilde{k}_0^M = \begin{cases} K+1, & \gamma = 1, \\ 1, & 0 < \gamma < 1. \end{cases}$$

Proof. (1) From (2.14), (2.4) and (3.5),

$$\begin{aligned}
\tilde{k}_n^M &= \langle Q_n(x), Q_n(x)\rangle_{\mathrm{S}}^{\mathrm{M}} = \langle Q_n(x), M_n^{(\gamma-1,\mu)}(x)\rangle_{\mathrm{S}}^{\mathrm{M}} \\
&= \langle \boldsymbol{u}_0, Q_n(x)M_n^{(\gamma-1,\mu)}(x)\rangle + \lambda n^2 k_{n-1}^{(\gamma,\mu)} \\
&= \langle \boldsymbol{u}^{(\gamma,\mu)}, Q_n(x)M_n^{(\gamma-1,\mu)}(x)\rangle + \lambda n^2 k_{n-1}^{(\gamma,\mu)} + \frac{(1-\gamma+a)(1-\mu)}{\gamma-1}k_n^{(\gamma-1,\mu)} \\
&= \left\langle \boldsymbol{u}^{(\gamma,\mu)}, \left(M_n^{(\gamma,\mu)}(x) + n\frac{\mu}{1-\mu}\left(M_{n-1}^{(\gamma,\mu)}(x) - \frac{k_{n-1}^{(\gamma,\mu)}}{\tilde{k}_{n-1}^M}Q_{n-1}(x)\right)\right)\right. \\
&\qquad \left.\times\left(M_n^{(\gamma,\mu)}(x) + n\frac{\mu}{1-\mu}M_{n-1}^{(\gamma,\mu)}(x)\right)\right\rangle \\
&\qquad + \lambda n^2 k_{n-1}^{(\gamma,\mu)} + \frac{(1-\gamma+a)(1-\mu)}{\gamma-1}k_n^{(\gamma-1,\mu)}.
\end{aligned}$$

Thus, (3.6) follows.

(2) If $0 < \gamma < 1$, (3.7) was obtained in [3]. In the case $\gamma = 1$, it is enough to take into account the relation (2.9) and use the same technique as in [3].

The initial condition can be deduced in the three above cases from the definition of the Sobolev inner product. □

Remark 1. Note that case $\gamma = 1$ in (3.7) is a consequence of (3.6) taking into account (2.10) as a limit case, since $\lim_{\gamma\downarrow 1} k_n^{(\gamma-1,\mu)} = 0$, $n \geqslant 1$.

Now, we obtain the asymptotic behavior of the squared Δ-Sobolev norms.

Lemma 3.4.

$$\eta = \lim_{n\to\infty} \frac{\tilde{k}_n^M}{k_n^{(\gamma,\mu)}} = \frac{\mu(1+\mu)+\lambda(1-\mu)^2+\sqrt{(\mu(1+\mu)+\lambda(1-\mu)^2)^2-4\mu^3}}{2\mu} > 1. \tag{3.8}$$

Proof. First, we assume $\gamma > 1$. If we divide (3.6) by $k_n^{(\gamma,\mu)}$, by using (2.11) we get

$$\frac{\tilde{k}_n^M}{k_n^{(\gamma,\mu)}} = A(n) + B(n) - C(n)\frac{k_{n-1}^{(\gamma,\mu)}}{\tilde{k}_{n-1}^M},$$

where

$$A(n) = 1 + \frac{(\lambda(\mu-1)^2+\mu^2)n}{\mu(\gamma+n-1)}, \quad B(n) = \frac{(1-\gamma+a)(1-\mu)}{\gamma+n-1}, \quad C(n) = \frac{\mu n}{n+\gamma-1}.$$

Let us define

$$s_{n+1} := s_n \frac{\tilde{k}_n^M}{k_n^{(\gamma,\mu)}}$$

with the initial condition $s_0 = 1$. Thus, we can write the above expression as

$$s_{n+1} = (A(n)+B(n))s_n - C(n)s_{n-1}, \tag{3.9}$$

where $s_0 = 1$ and $s_1 = \tilde{k}_0^M / k_0^{(\gamma,\mu)}$. Taking into account

$$\lim_{n\to\infty} A(n) = \frac{\mu^2+\mu+\lambda(1-\mu)^2}{\mu}, \quad \lim_{n\to\infty} B(n) = 0, \quad \lim_{n\to\infty} C(n) = \mu,$$

the roots of the limit characteristic equation of (3.9)

$$z^2 - \frac{\mu^2+\mu+\lambda(1-\mu)^2}{\mu} z + \mu = 0$$

are

$$z_1 = \frac{\mu(1+\mu)+\lambda(1-\mu)^2+\sqrt{(\mu(1+\mu)+\lambda(1-\mu)^2)^2-4\mu^3}}{2\mu},$$

$$z_2 = \frac{\mu(1+\mu)+\lambda(1-\mu)^2-\sqrt{(\mu(1+\mu)+\lambda(1-\mu)^2)^2-4\mu^3}}{2\mu}.$$

Because of Poincaré's Theorem, the sequence $\tilde{k}_n^M / k_n^{(\gamma,\mu)} = s_{n+1}/s_n$ converges to z_1 or z_2. On the other hand, using the extremal property of the monic polynomials and the expression of $\boldsymbol{u}_0$ given in (2.4) we get

$$\tilde{k}_n^M = \langle Q_n(x), Q_n(x)\rangle_{\mathrm{S}}^{\mathrm{M}} \geqslant \frac{1-\mu}{\gamma-1}\langle \boldsymbol{u}^{(\gamma-1,\mu)}, (x+a)R_n^2(x)\rangle + \lambda n^2 k_{n-1}^{(\gamma,\mu)},$$

where R_n is the monic polynomial of degree n, orthogonal with respect to the positive definite functional $[(\gamma-1)/(1-\mu)]\boldsymbol{u}_0$. Taking into account (see [10, p. 35])

$$(x+a)R_n(x) = M_{n+1}^{(\gamma-1,\mu)}(x) - \frac{M_{n+1}^{(\gamma-1,\mu)}(-a)}{M_n^{(\gamma-1,\mu)}(-a)} M_n^{(\gamma-1,\mu)}(x),$$

and so

$$\langle \boldsymbol{u}^{(\gamma-1,\mu)}, (x+a)R_n^2(x)\rangle = -\frac{M_{n+1}^{(\gamma-1,\mu)}(-a)}{M_n^{(\gamma-1,\mu)}(-a)}\, k_n^{(\gamma-1,\mu)}.$$

Thus,

$$\tilde{k}_n^M \geqslant \lambda n^2 k_{n-1}^{(\gamma,\mu)} - \frac{M_{n+1}^{(\gamma-1,\mu)}(-a)}{M_n^{(\gamma-1,\mu)}(-a)}\, \frac{1-\mu}{\gamma-1} k_n^{(\gamma-1,\mu)}.$$

Now, using (2.10) and (2.11) we get

$$\frac{\tilde{k}_n^M}{k_n^{(\gamma,\mu)}} \geqslant \frac{n}{n+\gamma-1}\left(\lambda\,\frac{(1-\mu)^2}{\mu} - (1-\mu)\,\frac{M_{n+1}^{(\gamma-1,\mu)}(-a)}{nM_n^{(\gamma-1,\mu)}(-a)}\right).$$

Taking into account the ratio asymptotic (2.15), we get that the sequence $\tilde{k}_n^M/k_n^{(\gamma,\mu)}$ is bounded from below by a sequence which converges to $1+\lambda(1-\mu)^2/\mu$. This means, taking into account $\tilde{k}_n^M/k_n^{(\gamma,\mu)}$ converges, that $\lim_{n\to\infty} \tilde{k}_n^M/k_n^{(\gamma,\mu)} = z_1 > 1$.

When $0<\gamma\leqslant 1$, we can divide (3.7) by $k_n^{(\gamma,\mu)}$ and we get

$$\frac{\tilde{k}_n^M}{k_n^{(\gamma,\mu)}} = A(n) - C(n)\frac{k_{n-1}^{(\gamma,\mu)}}{\tilde{k}_{n-1}^M}.$$

Following the same reasoning as in the case $\gamma>1$, we can prove the result in a more easy way because now for each $n\geqslant 0$, $\tilde{k}_n^M \geqslant k_n^{(\gamma,\mu)}$, that is, $\tilde{k}_n^M/k_n^{(\gamma,\mu)}\geqslant 1$. □

Now we can prove Theorem 3.1.

Proof of Theorem 3.1. If we divide (3.5) by $M_n^{(\gamma,\mu)}(x)$ we obtain

$$1 + n\frac{\mu}{1-\mu}\frac{M_{n-1}^{(\gamma,\mu)}(x)}{M_n^{(\gamma,\mu)}(x)} = C_n(x) + n\frac{\mu}{1-\mu}\frac{k_{n-1}^{(\gamma,\mu)}}{\tilde{k}_{n-1}^M}\frac{M_{n-1}^{(\gamma,\mu)}(x)}{M_n^{(\gamma,\mu)}(x)} C_{n-1}(x), \quad n\geqslant 1, \tag{3.10}$$

where

$$C_n(x) := \frac{Q_n(x)}{M_n^{(\gamma,\mu)}(x)}, \quad n\geqslant 0.$$

From (2.15) and Lemma 3.4

$$\lim_{n\to\infty} n\frac{\mu}{1-\mu}\frac{k_{n-1}^{(\gamma,\mu)}}{\tilde{k}_{n-1}^{M}}\frac{M_{n-1}^{(\gamma,\mu)}(x)}{M_n^{(\gamma,\mu)}(x)} = -\frac{\mu}{\eta},$$

holds uniformly on compact subsets of $\mathbb{C}\backslash[0,\infty)$.

Now, we are in the same conditions as in the proof of Theorem 6 in [5]. Therefore, we can deduce the result in the same way as in [5]. $\square$

If we want to obtain a more detailed asymptotic information about the Δ-Sobolev polynomials, we must give the Plancherel–Rotach type asymptotics of these polynomials.

Theorem 3.5 (*Relative Plancherel–Rotach-type asymptotics*). *It holds*

$$\lim_{n\to\infty}\frac{Q_n(nx)}{M_n^{(\gamma,\mu)}(nx)} = \frac{\eta[\varphi(\frac{(1-\mu)x-(1+\mu)}{2\sqrt{\mu}}) + \sqrt{\mu}]}{\eta\varphi(\frac{(1-\mu)x-(1+\mu)}{2\sqrt{\mu}}) + \sqrt{\mu}}, \tag{3.11}$$

uniformly on compact subsets of $\mathbb{C}\backslash[0,(1+\sqrt{\mu})^2/(1-\mu)]$, *where* $\varphi(x)=x+\sqrt{x^2-1}$ *with* $\sqrt{x^2-1}>0$ *if* $x>1$, *i.e., the conformal mapping of* $\mathbb{C}\backslash[-1,1]$ *onto the exterior of the closed unit disk.*

Proof. Making the change of variable $x\to nx$ in (3.5), using this relation for the scaled polynomials in a recursive way and dividing by $M_n^{(\gamma,\mu)}(nx)$ we get

$$\frac{Q_n(nx)}{M_n^{(\gamma,\mu)}(nx)} = \sum_{j=0}^{n}(-1)^j b_{n-j}^{(n)}\frac{M_{n-j}^{(\gamma,\mu)}(nx) + (n-j)\frac{\mu}{1-\mu}M_{n-j-1}^{(\gamma,\mu)}(nx)}{M_n^{(\gamma,\mu)}(nx)}, \tag{3.12}$$

where

$$b_n^{(n)} = 1,$$

$$b_{n-j}^{(n)} = \left(\frac{\mu}{1-\mu}\right)^j\prod_{i=1}^{j}(n-i+1)\frac{k_{n-i}^{(\gamma,\mu)}}{\tilde{k}_{n-i}^{M}}, \quad j=1,\dots,n.$$

Now, we are in a similar situation as in the proof of Theorem 7 in [5] and like for that proof, we need again a dominant for (3.12) in order to apply Lebesgue's dominated convergence theorem. The key to the proof of Theorem 7 in [5] is that the sequence $k_n^{(\gamma,\mu)}/\tilde{k}_n^{M}\leqslant\mathscr{C}<1$ for all $n\geqslant 1$ (see [5, f.(14)]), but in general this is not true in our situation as we have observed by numerical computations for certain values of $\gamma>1$ and n being small. However, in [1] a technical result was established in order to solve a similar problem. This result can be rewritten as

Lemma 3.6. *There exist constants C and r with $C>1$ and $0<r<1$ such that* $d_i^{(n)} = k_{n-i}^{(\gamma,\mu)}/\tilde{k}_{n-i}^{M}$ *verify* $0<d_i^{(n)}<Cr^i$ *for all* $n\geqslant 0$ *and* $0\leqslant i\leqslant n$.

Proof. The proof is the same as the one of the Lemma 3.2 in [1] but now taking into account Lemma 3.4. $\square$

Using the above lemma, we can obtain a dominant for (3.12) and therefore, we get Theorem 3.5 following the same steps as in the proof of Theorem 7 in [5]. □

Obviously, the Corollaries 8 and 9 in [5] remain true for the Δ-Meixner–Sobolev orthogonal polynomials associated to Δ-coherent pairs of type I.

4. Laguerre–Sobolev as limit case of Meixner–Sobolev

In [15], the authors obtained asymptotic properties for coherent pairs of positive-definite linear functionals of Laguerre type. In this section, we shall recover some of these results using another approach via limit relations by using the asymptotic properties for Δ-coherent pairs of positive-definite linear functionals of Meixner type obtained in the previous section.

Monic Laguerre polynomials $\{L_n^{(\alpha)}(x)\}_n$ are orthogonal with respect to the Laguerre linear functional $\boldsymbol{u}^{(\alpha)}$ defined by

$$\langle \boldsymbol{u}^{(\alpha)}, p\rangle := \int_0^{+\infty} p(x)\frac{x^\alpha \mathrm{e}^{-x}}{\Gamma(\alpha+1)}\,\mathrm{d}x, \quad \alpha > -1, \quad p \in \mathbb{P}.$$

We shall denote

$$k_n^{(\alpha)} := \langle \boldsymbol{u}^{(\alpha)}, (L_n^{(\alpha)}(x))^2\rangle = n!\frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)}, \quad n \geqslant 0. \tag{4.1}$$

Note that

$$\lim_{\mu\uparrow 1}(1-\mu)^{2n} k_n^{(\alpha+1,\mu)} = k_n^{(\alpha)}, \quad n \geqslant 0 \tag{4.2}$$

as well as

$$\lim_{\mu\uparrow 1}(1-\mu)^{n} M_n^{(\alpha+1,\mu)}\left(\frac{x}{1-\mu}\right) = L_n^{(\alpha)}(x), \quad n \geqslant 0. \tag{4.3}$$

In [14] Meijer obtained

Theorem 4.1. *Let $(\boldsymbol{u}_0, \boldsymbol{u}_1)$ be a coherent pair of positive-definite linear functionals of Laguerre type. If $\boldsymbol{u}_1 = \boldsymbol{u}^{(\alpha)}$ is the Laguerre linear functional, then*

(a) *If $\alpha > 0$, then*

$$\boldsymbol{u}_0 = \frac{(x+a)}{\alpha}\boldsymbol{u}^{(\alpha-1)}, \quad a \geqslant 0.$$

(b) *If $\alpha = 0$, then $\boldsymbol{u}_0 = \boldsymbol{u}^{(0)} + M\boldsymbol{\delta}(x)$, where $M \geqslant 0$.*

(c) *If $-1 < \alpha < 0$, then $\boldsymbol{u}_0 = \boldsymbol{u}^{(\alpha)}$.*

We shall refer to a coherent pair of linear functionals, where $\boldsymbol{u}_1$ is the Laguerre linear functional as a coherent pair of linear functionals of Laguerre type I.

Let $(\boldsymbol{u}_0, \boldsymbol{u}_1)$ be a coherent pair of positive-definite linear functionals of Laguerre type I, and let us denote

$$\langle p, r\rangle_{\mathrm{S}}^{\mathrm{L}} := \langle \boldsymbol{u}_0, pr\rangle + \lambda\langle \boldsymbol{u}^{(\alpha)}, p'r'\rangle, \quad \lambda \geqslant 0. \tag{4.4}$$

Let $\{Q_n(x;\alpha,\lambda,M,a)\}_n$ be the MOPS associated with the above inner product. We shall denote

$$\tilde{k}_n^L = \langle Q_n(x;\alpha,\lambda,M,a), Q_n(x;\alpha,\lambda,M,a)\rangle_{\mathrm{S}}^{\mathrm{L}}. \tag{4.5}$$

Lemma 4.2 (*Meijer et al., Lemma 3.1*). *Let $(\boldsymbol{u}_0, \boldsymbol{u}_1)$ be a coherent pair of positive-definite linear functionals of Laguerre type I. Then,*

$$L_n^{(\alpha)}(x) + nL_{n-1}^{(\alpha)}(x) = Q_n(x;\alpha,\lambda,M,a) + n\frac{k_{n-1}^{(\alpha)}}{\tilde{k}_{n-1}^L}Q_{n-1}(x;\alpha,\lambda,M,a), \quad n \geqslant 1, \tag{4.6}$$

where $k_n^{(\alpha)}$ and $\tilde{k}_n^L$ are given in (4.1) *and* (4.5), *respectively, and* $Q_0(x;\alpha,\lambda,M,a) = 1$.

Lemma 4.3 (*Meijer et al., Lemma 3.2*). (1) *If $\alpha > 0$, then*

$$\tilde{k}_n^L = k_n^{(\alpha)} + (\lambda+1)n^2k_{n-1}^{(\alpha)} + \frac{a}{\alpha}k_n^{(\alpha-1)} - n^2\frac{(k_{n-1}^{(\alpha)})^2}{\tilde{k}_{n-1}^L}, \quad n \geqslant 1, \quad \tilde{k}_0^L := \frac{\alpha+a}{\alpha}. \tag{4.7}$$

(2) *If $-1 < \alpha \leqslant 0$, then*

$$\tilde{k}_n^L = k_n^{(\alpha)} + (\lambda+1)n^2k_{n-1}^{(\alpha)} - n^2\frac{(k_{n-1}^{(\alpha)})^2}{\tilde{k}_{n-1}^L}, \quad n \geqslant 1, \quad \tilde{k}_0^L := \begin{cases} 1, & -1 < \alpha < 0, \\ 1+M, & \alpha = 0. \end{cases} \tag{4.8}$$

Proposition 4.4. *Let $\{Q_n(x;\alpha;\lambda,M,a)\}_n$ be the MOPS associated with the inner product* (4.4) *and let $\{Q_n(x;\gamma,\mu;\lambda,M,a)\}_n$ be the MOPS associated with the inner product* (3.1). *Then,*

$$\lim_{\mu\uparrow 1}(1-\mu)^n Q_n\left(\frac{x}{1-\mu};\alpha+1,\mu;\frac{\lambda}{(1-\mu)^2},M,\frac{a}{1-\mu}\right) = Q_n(x;\alpha;\lambda,M,a). \tag{4.9}$$

Proof. If $\alpha > 0$, first note that the following limit holds:

$$\lim_{\mu\uparrow 1}(1-\mu)^{2n}\tilde{k}_n^M\left(\alpha+1,\mu;\frac{\lambda}{(1-\mu)^2},\frac{a}{1-\mu}\right) = \tilde{k}_n^L(\alpha,\lambda,a) \tag{4.10}$$

as consequence of

$$\lim_{\mu\uparrow 1}\tilde{k}_0^M\left(\alpha+1,\mu;\frac{\lambda}{(1-\mu)^2},\frac{a}{1-\mu}\right) = \tilde{k}_0^L(\alpha,\lambda,a) = 1 + \frac{a}{\alpha},$$

and (4.2). The limit relation (4.9) follows from (3.5), (4.3), (4.10) and

$$Q_0\left(\frac{x}{1-\mu};\alpha+1,\mu;\frac{\lambda}{(1-\mu)^2},M,\frac{a}{1-\mu}\right) = Q_0(x;\gamma,\mu;\lambda,M,a) = 1.$$

If $-1 < \alpha \leqslant 0$, the proof follows as in the previous case from (4.8). □

Corollary 4.5. *Let* $\{L_n^{(\alpha)}(x)\}_n$ *be the Laguerre MOPS and let* $\{Q_n(x;\alpha,\lambda,M,a)\}_n$ *be the MOPS associated with* (4.4). *Then,*

$$\lim_{n\to\infty} \frac{Q_n(x;\alpha,\lambda,M,a)}{L_n^{(\alpha-1)}(x)} = \frac{\lambda+\sqrt{\lambda(4+\lambda)}}{2\lambda}, \tag{4.11}$$

uniformly on compact subsets of $\mathbb{C}\backslash[0,+\infty)$.

Proof. In order to recover (4.11) from the asymptotic properties of Meixner type, we can compute directly the limit as $\mu \uparrow 1$ in (3.4), with $\lambda \to \lambda/(1-\mu)^2$ (see 4.9), since all the steps given in order to obtain the asymptotic behavior (3.4) for Δ-coherent pairs of linear functionals of Meixner type I, are valid if we first compute an appropriate limit when $\mu \uparrow 1$. $\square$

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References

[1] M. Alfaro, J.J. Moreno-Balcázar, M.L. Rezola, Laguerre–Sobolev orthogonal polynomials: asymptotics for coherent pairs of type II, J. Approx. Theory 122 (2003) 79–96.
[2] I. Area, E. Godoy, F. Marcellán, Classification of all Δ-coherent pairs, Integral Transforms Special Functions 9 (1) (2000) 1–18.
[3] I. Area, E. Godoy, F. Marcellán, Inner products involving differences: the Meixner–Sobolev polynomials, J. Difference Equations Appl. 6 (2000) 1–31.
[4] I. Area, E. Godoy, F. Marcellán, Δ-coherent pairs and orthogonal polynomials of a discrete variable, Integral Transforms Special Functions 14 (1) (2003) 31–57.
[5] I. Area, E. Godoy, F. Marcellán, J.J. Moreno-Balcázar, Ratio and Plancherel–Rotach asymptotics for Meixner–Sobolev polynomials, J. Comput. Appl. Math. 116 (1) (1999) 63–75.
[6] I. Area, E. Godoy, A. Ronveaux, A. Zarzo, Minimal recurrence relations for connection coefficients between classical orthogonal polynomials: discrete case, J. Comput. Appl. Math. 89 (2) (1998) 309–325.
[7] H. Bavinck, On polynomials orthogonal with respect to an inner product involving differences, J. Comput. Appl. Math. 57 (1995) 17–27.
[8] H. Bavinck, On polynomials orthogonal with respect to an inner product involving differences (The general case), Appl. Anal. 59 (1995) 233–240.
[9] H. Bavinck, R. Koekoek, Difference operators with Sobolev-type Meixner polynomials as eigenfunctions, Comput. Math. Appl. 36 (10–12) (1998) 163–177.
[10] T.S. Chihara, An Introduction to Orthogonal Polynomials, Gordon and Breach, New York, 1978.
[11] A.G. García, F. Marcellán, L. Salto, A distributional study of discrete classical orthogonal polynomials, J. Comput. Appl. Math. 57 (1995) 147–162.
[12] A. Iserles, P.E. Koch, S.P. Nørsett, J.M. Sanz-Serna, On polynomials orthogonal with respect to certain Sobolev inner products, J. Approx. Theory 65 (2) (1991) 151–175.

[13] A. Martínez-Finkelshtein, Analytic aspects of Sobolev orthogonality revisited, J. Comput. Appl. Math. 127 (2001) 255–266.
[14] H.G. Meijer, Determination of all coherent pairs, J. Approx. Theory 89 (3) (1997) 321–343.
[15] H.G. Meijer, T.E. Pérez, M.A. Piñar, Asymptotics of Sobolev orthogonal polynomials for coherent pairs of Laguerre type, J. Math. Anal. Appl. 245 (2) (2000) 528–546.
[16] A.F. Nikiforov, S.K. Suslov, V.B. Uvarov, Classical Orthogonal Polynomials of a Discrete Variable, Springer, Berlin, 1991.